

FULL GROUPS OF ONE-SIDED TOPOLOGICAL MARKOV SHIFTS

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ABSTRACT. Let (X_A, σ_A) be the right one-sided topological Markov shift for an irreducible matrix with entries in $\{0, 1\}$, and Γ_A the continuous full group of (X_A, σ_A) . For two irreducible matrices A and B with entries in $\{0, 1\}$, it will be proved that the continuous full groups Γ_A and Γ_B are isomorphic as abstract groups if and only if their one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.

1. INTRODUCTION

Giordano-Putnam-Skau have introduced studies of orbit equivalences for minimal homeomorphisms on Cantor sets ([9], [10], cf. [8], [12], [18], etc.). Minimal homeomorphisms on Cantor sets are now called Cantor minimal systems. In their theory, the full groups play crucial rôle to classify Cantor minimal systems under orbit equivalences. The full groups are defined as groups of homeomorphisms of the Cantor sets whose orbits are contained in the orbits of the minimal homeomorphisms. Giordano-Putnam-Skau have proved that the full groups as groups are complete invariants for orbit equivalences of Cantor minimal systems ([10]). (For measure theoretic studies for orbit equivalences of ergodic transformations, see [3], [6], [7], [11], [13], etc.).

The class of topological Markov shifts is another important class of topological dynamical systems on Cantor sets. The author has introduced a study of orbit equivalence of one-sided topological Markov shifts, related to classification of Cuntz-Krieger algebras ([15], [16]). Let A be an $N \times N$ irreducible matrix with entries in $\{0, 1\}$ satisfying condition (I) in the sense of Cuntz-Krieger [5]. Let us denote by X_A the shift space

$$X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, \dots, N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}$$

over $\{1, \dots, N\}$. It is homeomorphic to a Cantor set in natural product topology. The continuous surjective map σ_A on X_A is defined by $\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$. The topological dynamical system (X_A, σ_A) is called the (right) one-sided topological Markov shift for A . The continuous map σ_A on X_A is no longer homeomorphism and has dense periodic points. Hence the one-sided topological Markov shifts are considered to locate far from Cantor minimal systems in the topological dynamical systems on Cantor sets. The continuous full group Γ_A of (X_A, σ_A) is defined to be the group of homeomorphisms τ on X_A satisfying

$$\tau(x) \in \text{orb}_{\sigma_A}(x) = \cup_{k=0}^{\infty} \cup_{l=0}^{\infty} \sigma_A^{-k}(\sigma_A^l(x)) \quad \text{for all } x \in X_A$$

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and having continuous orbit cocycles.

In this paper, we will study the continuous full groups on one-sided topological Markov shifts by using techniques in [10] for analyzing full groups of Cantor minimal systems. Keeping in mind the results of the Giordano-Putnam-Skau's paper [10] on the Cantor minimal systems, the algebraic structure of the continuous full groups are expected to determine the structure of continuous orbit equivalences of one-sided topological Markov shifts. Let A, B be two irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I). If there exists a homeomorphism $h : X_A \rightarrow X_B$ such that $h \circ \Gamma_A \circ h^{-1} = \Gamma_B$, then the continuous full groups Γ_A and Γ_B on X_B are said to be spatially isomorphic. We have proved in [15] that Γ_A and Γ_B are spatially isomorphic if and only if the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent. We will prove in this paper that an algebraic isomorphism between the continuous full groups of one-sided topological Markov shifts yields an spatial isomorphism between them. Hence the algebraic structure of the continuous full groups of one-sided topological Markov shifts are complete invariants of continuous orbit equivalence classes of one-sided topological Markov shifts. This is an analogue of the Giordano-Putnam-Skau's theorem [10, Theorem 4.2] for one-sided topological Markov shifts. The strategy to prove it basically follows Giordano-Putnam-Skau's paper [10] in which the algebraic isomorphism between the full groups of Cantor minimal systems induces a spatial isomorphism between them ([10, Theorem 4.2]). The continuous full groups of one-sided topological Markov shifts are countable, nonamenable groups (cf. [17]). They are huge groups rather than the full groups of Cantor minimal systems. Indeed the full groups of Cantor minimal systems have invariant probability measures, whereas the continuous full groups of one-sided topological Markov shifts do not have any invariant probability measure. In following the proof of [10, Theorem 4.2], there are several places where the proofs of [10, Theorem 4.2] do not well work in our setting because our full groups are huge. We may modify their proofs which we apply to our situations and will reach our goal stated as the following theorem:

Theorem 1.1. *Let A, B be two irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I). Then the following two conditions are equivalent:*

- (i) *The continuous full groups Γ_A and Γ_B are isomorphic as abstract groups.*
- (ii) *The continuous full groups Γ_A and Γ_B are spatially isomorphic.*

As (ii) \implies (i) is clear, the implication (i) \implies (ii) is the main part. For an irreducible square matrix A with entries in $\{0, 1\}$, denote by \mathcal{O}_A its Cuntz-Krieger algebra. We also denote by \mathcal{D}_A its canonical maximal abelian subalgebra of \mathcal{O}_A . The class of Cuntz-Krieger algebras plays an important rôle in classification theory of simple purely infinite C^* -algebras. Related to the classification of the Cuntz-Krieger algebras, the author has shown that there exists an isomorphism $\Psi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ satisfying $\Psi(\mathcal{D}_A) = \mathcal{D}_B$ if and only if Γ_A and Γ_B are spatially isomorphic, which is also equivalent to the condition that the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent ([15]). Therefore we know

Corollary 1.2. *Let A, B be two irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I). Then the following three conditions are equivalent:*

- (i) *The continuous full groups Γ_A and Γ_B are isomorphic as abstract groups.*

- (ii) *The one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*
- (iii) *There exists an isomorphism $\Psi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ satisfying $\Psi(\mathcal{D}_A) = \mathcal{D}_B$.*

Let N and M be the size of matrix A and that of B respectively. Denote by I_N and by I_M the identity matrix of size N and that of size M respectively. In [16], under the condition that $\det(A - I_N)\det(B - I_M) \geq 0$, an isomorphism between Cuntz-Krieger algebras induces an isomorphism between them which preserves their canonical maximal abelian subalgebras. Hence we have

Corollary 1.3. *Let A, B be two irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I). Suppose that $\det(A - I_N)\det(B - I_M) \geq 0$. Then the continuous full groups Γ_A and Γ_B are isomorphic as abstract groups if and only if the Cuntz-Krieger algebras \mathcal{O}_A and \mathcal{O}_B are isomorphic.*

By using classification theorem for Cuntz-Krieger algebras obtained by M. Rørdam [20](cf. [21]), one may classify the continuous full groups in terms of the underlying matrices under the determinant condition $\det(A - I_N)\det(B - I_M) \geq 0$ as follows:

Corollary 1.4. *Let A, B be two irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I). Suppose that $\det(A - I_N)\det(B - I_M) \geq 0$. The continuous full groups Γ_A and Γ_B are isomorphic as abstract groups if and only if there exists an isomorphism $\Phi : \mathbb{Z}^N / (A^t - I_N)\mathbb{Z}^N \rightarrow \mathbb{Z}^M / (B^t - I_M)\mathbb{Z}^M$ such that $\Phi([1, \dots, 1]) = [1, \dots, 1]$.*

Therefore we know that there are many mutually nonisomorphic continuous full groups of one-sided topological Markov shifts such as the following corollary.

Corollary 1.5. *Let N, M be positive integers such that $N, M > 1$. Denote by $\Gamma_{[N]}$ and $\Gamma_{[M]}$ the continuous full groups of the one-sided full N -shift and M -shift respectively. Then $\Gamma_{[N]}$ and $\Gamma_{[M]}$ are isomorphic as abstract groups if and only if $N = M$.*

The main part of the paper is devoted to proving the implication (i) \implies (ii) of Theorem 1.1. The paper is organized to prove it as in the following way. In Section 2, some basic properties of continuous full groups will be stated. In Section 3, an open set of X_A will be described in terms of a pair, called a strong commuting pair, of subgroups of Γ_A . In Section 4, a condition under which an open set of X_A becomes clopen will be described in terms of algebraic conditions of a strong commuting pair. In Section 5, support of a strong commuting pair will be defined and proved to be clopen. In Section 6, an clopen set of X_A will be completely replaced in terms of a pair of subgroups of Γ_A with some algebraic conditions. The pair of subgroups are called Dye pairs. In Section 7, the main result and its corollaries will be stated. The above strategy to prove Theorem 1.1 basically follows [10]. However several proofs of in particular Lemma 4.6, Lemma 4.8, Lemma 4.11 and Lemma 6.3 are essentially different from the paper [10]. The discussions of Section 4 and Section 6 are tough parts in this paper.

2. THE CONTINUOUS FULL GROUPS AND THE LOCAL SUBGROUPS

Let $A = [A(i, j)]_{i, j=1}^N$ be an $N \times N$ matrix with entries in $\{0, 1\}$, where $1 < N \in \mathbb{N}$. The matrix A is always assumed to be essential, which means that it has no zero columns or zero rows. We assume that A is irreducible and satisfies condition

(I) in the sense of Cuntz-Krieger [5]. In what follows, we fix the matrix A . We denote by X_A the shift space

$$X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, \dots, N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}$$

over $\{1, \dots, N\}$ of the right one-sided topological Markov shift for A . It is a compact Hausdorff space in natural product topology. The condition (I) for A is equivalent to the condition that X_A is homeomorphic to a Cantor discontinuum. The shift transformation σ_A on X_A is defined by $\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$ for $(x_n)_{n \in \mathbb{N}}$. It is a continuous surjective map on X_A . The topological dynamical system (X_A, σ_A) is called the (right) one-sided topological Markov shift for A . A word $\mu = (\mu_1, \dots, \mu_k)$ for $\mu_i \in \{1, \dots, N\}$ is said to be admissible in X_A if μ appears in somewhere in some element x in X_A . The length of μ is k , which is denoted by $|\mu|$. We denote by $B_k(X_A)$ the set of all admissible words of length $k \in \mathbb{N}$. For $k = 0$ we denote by $B_0(X_A)$ the empty word \emptyset . We set $B_*(X_A) = \cup_{k=0}^{\infty} B_k(X_A)$ the set of admissible words of X_A . For $x = (x_n)_{n \in \mathbb{N}} \in X_A$ and positive integers k, l with $k \leq l$, we put the word $x_{[k, l]} = (x_k, x_{k+1}, \dots, x_l) \in B_{l-k+1}(X_A)$ and the right infinite sequence $x_{[k, \infty)} = (x_k, x_{k+1}, \dots) \in X_A$. We similarly use the notation $\mu_{[k, l]} = (\mu_k, \mu_{k+1}, \dots, \mu_l) \in B_{l-k+1}(X_A)$ for a word $\mu = (\mu_1, \dots, \mu_m) \in B_m(X_A)$ with $k \leq l \leq m$. For words $\mu = (\mu_1, \dots, \mu_k) \in B_k(X_A), \nu = (\nu_1, \dots, \nu_l) \in B_l(X_A)$ with $A(\mu_k, \nu_1) = 1$, denote by $\mu\nu$ its concatenation $\mu\nu = (\mu_1, \dots, \mu_k, \nu_1, \dots, \nu_l) \in B_{k+l}(X_A)$. For a word $\mu = (\mu_1, \dots, \mu_k) \in B_k(X_A)$, we denote by U_μ its cylinder set

$$U_\mu = \{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \dots, \mu_k = x_k\}.$$

For $x = (x_n)_{n \in \mathbb{N}} \in X_A$, the orbit $\text{orb}_{\sigma_A}(x)$ of x under σ_A is defined by

$$\text{orb}_{\sigma_A}(x) = \cup_{k=0}^{\infty} \cup_{l=0}^{\infty} \sigma_A^{-k}(\sigma_A^l(x)) \subset X_A.$$

Hence $y = (y_n)_{n \in \mathbb{N}} \in X_A$ belongs to $\text{orb}_{\sigma_A}(x)$ if and only if there exist $k, l \in \mathbb{Z}_+$ and an admissible word $(\mu_1, \dots, \mu_k) \in B_k(X_A)$ such that

$$y = (\mu_1, \dots, \mu_k, x_{l+1}, x_{l+2}, \dots).$$

We denote by $\text{Homeo}(X_A)$ the group of all homeomorphisms on X_A . We have defined in [15] the continuous full group Γ_A for (X_A, σ_A) as in the following way.

Definition ([15]). Let τ be a homeomorphism on X_A such that $\tau(x) \in \text{orb}_{\sigma_A}(x)$ for all $x \in X_A$. Hence there exist functions $k, l : X_A \rightarrow \mathbb{Z}_+$ such that

$$\sigma_A^{k(x)}(\tau(x)) = \sigma_A^{l(x)}(x) \quad \text{for all } x \in X_A. \quad (2.1)$$

Let Γ_A be the set of all homeomorphisms τ such that there exist continuous functions $k, l : X_A \rightarrow \mathbb{Z}_+$ satisfying (2.1). The set Γ_A is a subgroup of $\text{Homeo}(X_A)$ and is called the *continuous full group* of (X_A, σ_A) . The functions k, l above are called the orbit cocycles for τ . They are not necessarily uniquely determined by τ . We note that the group Γ_A has been written as $[\sigma_A]_c$ in the earlier papers [15], [16].

A continuous map $\tau : X_A \rightarrow X_A$ is called a *cylinder map* if there exist $L \in \mathbb{N}$ and $\Phi : B_L(X_A) \rightarrow B_*(X_A)$ such that $\Phi(\nu) = \mu(\nu) = (\mu_1(\nu), \dots, \mu_k(\nu)(\nu)) \in B_{k(\nu)}(X_A)$ for $\nu = \nu_1 \cdots \nu_L \in B_L(X_A)$ satisfies

$$\tau(\nu_1, \dots, \nu_L, x_{L+1}, x_{L+2}, \dots) = (\mu_1(\nu), \dots, \mu_k(\nu)(\nu), x_{L+1}, x_{L+2}, \dots)$$

for $(x_{L+1}, x_{L+2}, \dots) \in X_A$ with $A(\nu_L, x_{L+1}) = 1$. That is, τ satisfies $\tau(U_\nu) = U_{\Phi(\nu)}$ for $\nu \in B_L(X_A)$ with a map $\Phi : B_L(X_A) \rightarrow B_*(X_A)$. We see that a homeomorphism τ of X_A belongs to Γ_A if and only if τ is a cylinder map ([17]).

We will first study local structure of elements of Γ_A . Following [10], we will use the notations as follows:

$$\begin{aligned} O(X_A) &= \text{The set of all open sets of } X_A. \\ CL(X_A) &= \text{The set of all closed sets of } X_A. \\ CO(X_A) &= \text{The set of all clopen sets of } X_A. \end{aligned}$$

We note that an open set of X_A is a countable disjoint union of cylinder sets and a clopen set of X_A is a finite disjoint union of cylinder sets.

Lemma 2.1. *For nonempty open sets $U, Y \in O(X_A)$ and $x \in U$, there exist $V \in CO(X_A)$ and $\alpha \in \Gamma_A$ such that*

$$x \in V \subset U, \quad \alpha(V) \subset Y, \quad \alpha^2 = \text{id}, \quad \alpha|_{(V \cup \alpha(V))^c} = \text{id}.$$

Proof. Take a cylinder set U_μ for some word $\mu = (\mu_1, \dots, \mu_n) \in B_n(X_A)$ such that $U_\mu \subset Y$. Since A satisfies condition (I), there exist distinct words $s = (s_1, \dots, s_k), s' = (s'_1, \dots, s'_k) \in B_k(X_A)$ and $u \in \{1, \dots, N\}$ such that

$$A(\mu_n, s_1) = A(\mu_n, s'_1) = A(s_k, u) = A(s'_k, u) = 1.$$

There exists a word $\nu = (\nu_1, \dots, \nu_m) \in B_m(X_A)$ such that $m > n + k + 1$ and $x \in U_\nu \subset U$. Now A is irreducible so that one may find a word $\xi = (\xi_1, \dots, \xi_l) \in B_l(X_A)$ such that $A(u, \xi_1) = A(\xi_l, \nu_m) = 1$. Put

$$\begin{aligned} \bar{\mu} &= (\mu_1, \dots, \mu_n, s_1, \dots, s_k, u, \xi_1, \dots, \xi_l, \nu_m) \in B_{n+k+l+2}(X_A), \\ \bar{\mu}' &= (\mu_1, \dots, \mu_n, s'_1, \dots, s'_k, u, \xi_1, \dots, \xi_l, \nu_m) \in B_{n+k+l+2}(X_A). \end{aligned}$$

Since $\bar{\mu}_{[1, n+k+1]} \neq \bar{\mu}'_{[1, n+k+1]}$ and $|\nu| = m > n + k + 1$, at least either $\bar{\mu}_{[1, n+k+1]}$ or $\bar{\mu}'_{[1, n+k+1]}$ is different from $\nu_{[1, n+k+1]}$. We assume that $\bar{\mu}_{[1, n+k+1]} \neq \nu_{[1, n+k+1]}$, so that $U_\nu \cap U_{\bar{\mu}} = \emptyset$. Put $V = U_\nu$ and $L = n + k + l + 2$. Define $\alpha \in \Gamma_A$ by setting

$$\alpha(x) = \begin{cases} \bar{\mu}x_{[m+1, \infty)} & \text{if } x = \nu x_{[m+1, \infty)} \in U_\nu, \\ \nu x_{[L+1, \infty)} & \text{if } x = \bar{\mu}x_{[L+1, \infty)} \in U_{\bar{\mu}}, \\ x & \text{otherwise} \end{cases}$$

for $x \in X_A$. Then we have

$$\alpha(V) = U_{\bar{\mu}} \subset U_\mu \subset Y, \quad \alpha^2 = \text{id}, \quad \alpha|_{(U_\nu \cup U_{\bar{\mu}})^c} = \text{id}.$$

□

For clopen sets U, V of X_A , if there exists $\gamma \in \Gamma_A$ such that $\gamma(U) = V$, then U is said to be Γ_A -equivalent to V and written as $U \underset{\Gamma_A}{\sim} V$.

Lemma 2.2. *Let $U, V \in CO(X_A)$ be nonempty clopen sets such that $U \cap V = \emptyset$. If $U \underset{\Gamma_A}{\sim} V$, there exists $\alpha \in \Gamma_A$ such that*

$$\alpha(U) = V, \quad \alpha^2 = \text{id}, \quad \alpha|_{(U \cup V)^c} = \text{id}. \quad (2.2)$$

Proof. Let $\gamma \in \Gamma_A$ satisfy $\gamma(U) = V$. One may define $\alpha \in \Gamma_A$ by setting

$$\alpha(x) = \begin{cases} \gamma(x) & \text{if } x \in U, \\ \gamma^{-1}(x) & \text{if } x \in V, \\ x & \text{otherwise} \end{cases}$$

for $x \in X_A$. As both U and V are clopen, α defines an element of Γ_A which satisfies (2.2). \square

Lemma 2.3 (cf. [10, Lemma 3.4]). *For any $U \in CO(X_A)$ and $x \in U$, there exists $\alpha \in \Gamma_A$ such that*

$$\alpha(x) \neq x, \quad \alpha^2 = \text{id}, \quad \alpha|_{U^c} = \text{id}.$$

Proof. Take clopen sets $U_1, Y_1 \subset U$ such that $x \in U_1$ and $U_1 \cap Y_1 = \emptyset$. By Lemma 2.1, there exist a clopen set V_1 with $x \in V_1 \subset U_1$ and $\alpha \in \Gamma_A$ such that

$$\alpha(V_1) \subset Y_1, \quad \alpha^2 = \text{id}, \quad \alpha|_{(V_1 \cup \alpha(V_1))^c} = \text{id}.$$

Since $V_1 \cap Y_1 = \emptyset$ and $U_1 \cup Y_1 \subset U$, one has $\alpha(x) \neq x$ and $\alpha|_{U^c} = \text{id}$. \square

We follow the notations below from [10].

Definition. For an open set O of X_A , we set

$$\Gamma_O = \{\gamma \in \Gamma_A \mid \gamma(x) = x \text{ for all } x \in O^c\}.$$

A subgroup of Γ_A of the form Γ_U for $U \in CO(X_A)$ is called a *local subgroup* of Γ_A . We note that the notation Γ_A for matrix A is fixed and always denoting the continuous full group of (X_A, σ_A) . For a subgroup H of Γ_A the commutant of H will be denoted by H^\perp :

$$H^\perp = \{\xi \in \Gamma_A \mid \xi\gamma = \gamma\xi \text{ for all } \gamma \in H\}$$

which is a subgroup of Γ_A . The following proposition shows that the action of local subgroups Γ_O on the underlying space X_A is different from that of local subgroups of the full groups of Cantor minimal systems.

Proposition 2.4. *Let O be a nonempty open set of X_A . There is no regular Borel probability measure on O invariant under Γ_O .*

Proof. Suppose that there is a regular Borel probability measure μ on O such that $\mu \circ g = \mu$ for all $g \in \Gamma_O$. Take $\nu(i) \in B_*(X_A), i = 1, 2, \dots$ such that O is a disjoint union $\sqcup_{i=1}^\infty U_{\nu(i)}$ so that $1 = \mu(O) = \sum_{i=1}^\infty \mu(U_{\nu(i)})$. One may find a word $\nu \in B_*(X_A)$ such that $U_\nu \subset O$ and $\mu(U_\nu) > 0$. Take $x = (x_n)_{n \in \mathbb{N}} \in U_\nu$ such that $\mu(U_{x_{[1,n]}}) > 0$ for all $n \in \mathbb{N}$ where $x_{[1,n]} = (x_1, \dots, x_n)$. One may find $i > |\nu|$ and $k > 1$ such that $x_i = x_{i+k}$. Put $u = x_i$ and

$$\begin{aligned} \zeta &= (x_1, \dots, x_{i-1}), & \xi &= (x_{i+1}, \dots, x_{i+k-1}), \\ \bar{\zeta} &= (x_1, \dots, x_{i-1}, x_i), & \bar{\xi} &= (x_{i+1}, \dots, x_{i+k-1}, x_{i+k}) \end{aligned}$$

so that $\bar{\zeta}\bar{\xi} = x_{[1,i+k]}$. As $i-1 \geq |\nu|$, one has $U_{\bar{\zeta}\bar{\xi}} \subset U_\nu$ and $\mu(U_{\bar{\zeta}\bar{\xi}}) > 0$. Since the matrix A is irreducible and satisfies condition (I), there exists $\eta \in B_*(X_A)$ such that $|\eta| > 1, u\eta u \in B_*(X_A)$ and $U_{u\xi u} \cap U_{u\eta u} = \emptyset$. Put $\bar{\eta} = \eta u$. Hence $\bar{\zeta}\bar{\eta} \in B_*(X_A)$

and $U_{\bar{\zeta}\bar{\eta}} \subset U_\nu \subset O$. Define homeomorphisms ψ, φ on X_A by setting for $y \in X_A$

$$\psi(y) = \begin{cases} \bar{\zeta}\bar{\eta}z \in U_{\bar{\zeta}\bar{\eta}} & \text{if } y = \bar{\zeta}\bar{\xi}z \in U_{\bar{\zeta}\bar{\xi}}, \\ \bar{\zeta}\bar{\xi}z \in U_{\bar{\zeta}\bar{\xi}} & \text{if } y = \bar{\zeta}\bar{\eta}z \in U_{\bar{\zeta}\bar{\eta}}, \\ y & \text{otherwise,} \end{cases}$$

$$\varphi(y) = \begin{cases} \bar{\zeta}\bar{\eta}\bar{\xi}z \in U_{\bar{\zeta}\bar{\eta}\bar{\xi}} & \text{if } y = \bar{\zeta}\bar{\eta}\bar{\eta}z \in U_{\bar{\zeta}\bar{\eta}\bar{\eta}}, \\ \bar{\zeta}\bar{\xi}z \in U_{\bar{\zeta}\bar{\xi}} & \text{if } y = \bar{\zeta}\bar{\eta}\bar{\xi}z \in U_{\bar{\zeta}\bar{\eta}\bar{\xi}}, \\ \bar{\zeta}\bar{\eta}\bar{\eta}z \in U_{\bar{\zeta}\bar{\eta}\bar{\eta}} & \text{if } y = \bar{\zeta}\bar{\xi}z \in U_{\bar{\zeta}\bar{\xi}}, \\ y & \text{otherwise.} \end{cases}$$

Since $U_{\bar{\zeta}} \subset U_\nu$, one sees that

$$U_{\bar{\zeta}\bar{\eta}}, U_{\bar{\zeta}\bar{\xi}}U_{\bar{\zeta}\bar{\eta}\bar{\xi}}, U_{\bar{\zeta}\bar{\eta}\bar{\eta}} \subset U_\nu \subset O.$$

Hence we have $\psi, \varphi \in \Gamma_O$ and $\psi^2 = \varphi^3 = \text{id}$. We put $F = U_{\bar{\zeta}\bar{\xi}} \subset U_\nu$ so that $\mu(F) > 0$ and

$$\psi(F) = U_{\bar{\zeta}\bar{\eta}}, \quad \varphi(F) = U_{\bar{\zeta}\bar{\eta}\bar{\eta}}, \quad \varphi^2(F) = U_{\bar{\zeta}\bar{\eta}\bar{\xi}}.$$

Since

$$\varphi(F) \cup \varphi^2(F) \subset \psi(F), \quad \varphi(F) \cap \varphi^2(F) = \emptyset,$$

we have

$$\mu(\varphi(F)) + \mu(\varphi^2(F)) \leq \mu(\psi(F)).$$

By hypothesis, μ is Γ_O -invariant, we have $\mu(F) = 0$ a contradiction. Therefore we conclude that there is no Γ_O -invariant regular Borel probability measure on O . \square

By using the above proof, one may prove that the subgroup $\langle \psi, \varphi \rangle$ generated by ψ, φ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ and hence Γ_O contains the free group F_2 on two generators. We will not use this fact in the discussions henceforth so that we will not give its detailed proof (see [17]).

We note the following lemma.

Lemma 2.5. *For an open set O of X_A , we have $\gamma(O) = O$ for $\gamma \in \Gamma_O$.*

Proof. Take $\gamma \in \Gamma_O$. Suppose that there is $y \in O$ such that $\gamma(y) \in O^c$. As $\gamma|_{O^c} = \text{id}$, one sees that $\gamma(\gamma(y)) = \gamma(y)$ so that $\gamma(y) = y \in O$ a contradiction. Hence we have $\gamma(O) \subset O$ and similarly $\gamma^{-1}(O) \subset O$, so that $\gamma(O) = O$. \square

The following notations also follow from [10].

- (1) For $O \in O(X_A)$, put $O^\perp = (\overline{O})^c = (O^c)^\circ \in O(X_A)$.
- (2) For $F \in CL(X_A)$, put $F^\perp = \overline{(F^c)} = (F^\circ)^c \in CL(X_A)$.
- (3) $O \in O(X_A)$ is said to be regular if $(O^\perp)^\perp = O$.
- (4) $F \in CL(X_A)$ is said to be regular if $(F^\perp)^\perp = F$.

Denote by $RO(X_A)$ the set of regular open subsets of X_A . We note that an open set O of X_A is regular if and only if O^c is a regular closed set of X_A .

Remark.

- (i) For an open set O , we have $(O^\perp)^\perp = (((\overline{O})^c)^c)^\circ$. Hence O is regular if and only if $(\overline{O})^\circ = O$.
- (ii) For a closed set F , we have $(F^\perp)^\perp = \overline{((F^\circ)^c)^c}$. Hence F is regular if and only if $\overline{F^\circ} = F$.

We note the following lemma.

Lemma 2.6. *Let $O \in O(X_A)$ be an open set. Then O^\perp is a regular open set.*

Proof. As $(O^\perp)^\perp = (\overline{O})^\circ$, we have $((O^\perp)^\perp)^\perp = (\overline{(O^\perp)^\perp})^\circ \supset O^\perp$. The inclusion relation $(\overline{O})^\circ \supset O$ implies $((O^\perp)^\perp)^\perp = ((\overline{O})^\circ)^\perp = [(\overline{O})^\circ]^c \subset (\overline{O})^c = O^\perp$, so that we have $((O^\perp)^\perp)^\perp = O^\perp$. \square

Since Lemma 2.3 is the same statement as [10, Lemma 3.4], the lemma below holds by the same proof as the proof of [10, Lemma 3.9].

Lemma 2.7 (cf. [10, Lemma 3.9]). *For $O, O_1, O_2 \in O(X_A)$, we have*

- (i) $O_1 \subset O_2$ if and only if $\Gamma_{O_1} \subset \Gamma_{O_2}$.
- (ii) $\Gamma_O \cap \Gamma_{O^\perp} = \{\text{id}\}$.
- (iii) $(\Gamma_O)^\perp = \Gamma_{O^\perp}$ and $\Gamma_O \subset ((\Gamma_O)^\perp)^\perp$.
- (iv) $O \in RO(X_A)$ implies $\Gamma_O = ((\Gamma_O)^\perp)^\perp$.

3. STRONG COMMUTING PAIRS

In this section, we will find algebraic conditions of the pair $(\Gamma_O, \Gamma_{O^\perp})$ of subgroups of Γ_A for a regular open set $O \in O(X_A)$, and prove Proposition 3.3.

Following [10, Definition 3.10], we say that a pair (H, K) of subgroups of Γ_A is a *commuting pair* if the following condition called (D1) holds:

$$(D1) \quad H^\perp = K, \quad K^\perp = H, \quad H \cap K = \{\text{id}\}.$$

A commuting pair (H, K) is called a *strong commuting pair* if the following extra condition called (D2) holds:

(D2) For any nontrivial normal subgroup N of H (resp. of K), then $N^\perp = K$ (resp. $N^\perp = H$) holds.

We may see similar statements in [10] as Lemma 3.11 and Lemma 3.12 to the following two lemmas. The proofs of [10, Lemma 3.11] and [10, Lemma 3.12] need [10, Lemma 3.3]. In our setting, we do not have a corresponding lemma to [10, Lemma 3.3]. We give complete proofs for the following two lemmas for the sake of completeness.

Lemma 3.1. *Let $O \subset X_A$ be a nonempty open set and $\eta \in \Gamma_O$ with $\eta \neq \text{id}$. For a nonempty clopen set $U \subset O$, there exists $\gamma \in \Gamma_O$ such that $\gamma^{-1}\eta\gamma|_U \neq \text{id}$.*

Proof. The proof below is basically similar to [10, Lemma 3.11]. Take a nonempty clopen partition U_1, U_2 of U so that $U = U_1 \sqcup U_2$. Since $\eta \neq \text{id}$ and $\eta \in \Gamma_O$, there exists a clopen set $Y \subset X_A$ such that $Y \subset O$ and $\eta(Y) \cap Y = \emptyset$. One may take Y small enough such as $U_1 \setminus \eta(Y) \neq \emptyset$ and $U_2 \setminus Y \neq \emptyset$. By Lemma 2.1, there exist $\alpha \in \Gamma_A$ and a nonempty clopen set $U'_1 \subset U_1 \setminus \eta(Y)$ such that

$$\alpha(U'_1) \subset Y, \quad \alpha^2 = \text{id}, \quad \alpha|_{(U'_1 \cup \alpha(U'_1))^c} = \text{id}.$$

For $U_2 \setminus Y$ and $\eta(\alpha(U'_1)) \subset \eta(Y)$, Lemma 2.1 assures that there exist $\beta \in \Gamma_A$ and a nonempty clopen set $U'_2 \subset U_2 \setminus Y$ such that

$$\beta(U'_2) \subset \eta(\alpha(U'_1)), \quad \beta^2 = \text{id}, \quad \beta|_{(U'_2 \cup \beta(U'_2))^c} = \text{id}.$$

As $\alpha(U'_1) \cap U'_2 \subset Y \cap U'_2 = \emptyset$, $\beta(U'_2) \cap U'_1 \subset \eta(Y) \cap U'_1 = \emptyset$, one notes that

$$[U'_1 \cup \alpha(U'_1)] \cap [U'_2 \cup \beta(U'_2)] = \emptyset.$$

Define $\gamma \in \Gamma_A$ by setting

$$\gamma = \begin{cases} \alpha & \text{on } U'_1 \cup \alpha(U'_1), \\ \beta & \text{on } U'_2 \cup \beta(U'_2), \\ \text{id} & \text{elsewhere.} \end{cases}$$

As $U'_1 \subset U_1 \subset O$ and $\alpha(U'_1) \subset Y \subset O$, and also $U'_2 \subset U_2 \subset O$ and $\beta(U'_2) \subset \eta(Y) \subset O$, both $U'_1 \cup \alpha(U'_1)$ and $U'_2 \cup \beta(U'_2)$ are subsets of O . Hence γ belongs to Γ_O . Since $\beta(U'_2) \subset \eta(\alpha(U'_1))$, we have

$$\alpha\eta^{-1}\beta(U'_2) \subset \alpha^2(U'_1) = U'_1$$

so that

$$\gamma^{-1}\eta\gamma(\alpha\eta^{-1}\beta(U'_2)) = \gamma^{-1}\eta\alpha\eta^{-1}\beta(U'_2) = \gamma^{-1}\beta(U'_2) = U'_2.$$

Therefore we have $\gamma^{-1}\eta\gamma|_U \neq \text{id}$. \square

Lemma 3.2. *Let $O \subset X_A$ be a nonempty open set and $\eta \in \Gamma_O$ with $\eta \neq \text{id}$. For a nonempty clopen set $U \subset O$ and $\gamma \in \Gamma_A$ such that $\gamma(U) \subset O$ and $U \cap \gamma(U) = \emptyset$, there exist a nonempty clopen set $U_1 \subset U$ and $\psi \in \Gamma_O$ such that*

$$\gamma(\psi^{-1}\eta\psi)(U_1) \cap (\psi^{-1}\eta\psi)(\gamma(U_1)) = \emptyset.$$

Proof. As $U \cup \gamma(U) \subset O$, by taking a smaller clopen set $U' \subset U$ such as $U' \cap \gamma(U') \neq \emptyset$, one may assume that $U \cup \gamma(U)$ is a proper subset of O and take a nonempty clopen subset $\tilde{U} \subset O \setminus (U \cup \gamma(U))$. For the clopen set \tilde{U} and $\eta \in \Gamma_O$ with $\eta \neq \text{id}$, by applying the proof of the preceding lemma, we have a nontrivial clopen partition $\tilde{U}_1 \sqcup \tilde{U}_2 = \tilde{U}$, a nonempty clopen subset \tilde{U}'_2 of \tilde{U}_2 and $\tilde{\gamma} \in \Gamma_A$ such that

$$\tilde{\gamma}^{-1}\eta\tilde{\gamma}(\alpha\eta^{-1}\beta(\tilde{U}'_2)) \subset \tilde{U}'_2 \subset \tilde{U}_2.$$

Put $Y = \alpha\eta^{-1}\beta(\tilde{U}'_2) \subset \tilde{U}_1$ so that $\tilde{\gamma}^{-1}\eta\tilde{\gamma}(Y) \subset \tilde{U}_2$ and $\tilde{\gamma}^{-1}\eta\tilde{\gamma}(Y) \cap Y = \emptyset$. By putting $\tilde{\gamma}^{-1}\eta\tilde{\gamma}$ as η , one has

$$Y \cap \eta(Y) = \emptyset, \quad Y \cup \eta(Y) \subset O \setminus (U \cup \gamma(U)).$$

Let $U = U^{(1)} \sqcup U^{(2)} \sqcup U^{(3)}$ be a nontrivial clopen partition of U . By Lemma 2.1, there exist $\hat{\alpha} \in \Gamma_A$ with a clopen subset $\hat{U}_1 \subset U^{(1)}$ such that $\hat{\alpha}(\hat{U}_1) \subset U^{(2)}$ and $\hat{\beta} \in \Gamma_A$ with a clopen subset $\hat{U}_2 \subset \hat{\alpha}(\hat{U}_1)$ such that $\hat{\beta}(\hat{U}_2) \subset U^{(3)}$. Put

$$U' = \hat{\alpha}^{-1}(\hat{U}_2) \subset \hat{U}_1 \subset U^{(1)}, \quad U'' = \hat{U}_2 \subset U^{(2)}, \quad U''' = \hat{\beta}(\hat{U}_2) \subset U^{(3)}.$$

Hence we have a disjoint Γ_A -equivalent clopen partition:

$$U' \sqcup U'' \sqcup U''' \subset U.$$

Let $Y = Y_1 \sqcup Y_2$ be a nontrivial clopen partition of Y . By Lemma 2.1, there exist $U_1 \subset U'$ and $\alpha, \beta \in \Gamma_A$ such that

$$\alpha^2 = \beta^2 = \text{id}, \quad \alpha(U_1) \subset Y_1, \quad \alpha(\gamma(U_1)) \subset Y_2, \quad (3.1)$$

$$\beta(\eta\alpha(U_1)) \subset U'', \quad \beta(\eta\alpha\gamma(U_1)) \subset \gamma(U''') \quad (3.2)$$

and

$$\alpha|_{[U_1 \cup \alpha(U_1) \cup \gamma(U_1) \cup \alpha\gamma(U_1)]^c} = \text{id}, \quad (3.3)$$

$$\beta|_{[\eta\alpha(U_1) \cup \beta\eta\alpha(U_1) \cup \eta\alpha\gamma(U_1) \cup \beta\eta\alpha\gamma(U_1)]^c} = \text{id}. \quad (3.4)$$

In fact, by applying Lemma 2.1 for U' and Y_1 , we have $V_1 \subset U'$ and $\alpha_1 \in \Gamma_A$ such that

$$\alpha_1(V_1) \subset Y_1, \quad \alpha_1^2 = \text{id}, \quad \alpha_1|_{(V_1 \cup \alpha_1(V_1))^c} = \text{id}.$$

By applying Lemma 2.1 for $\gamma(V_1)$ and Y_2 , we have $V_1' \subset \gamma(V_1)$ and $\alpha_2 \in \Gamma_A$ such that

$$\alpha_2(V_1') \subset Y_2, \quad \alpha_2^2 = \text{id}, \quad \alpha_2|_{(V_1' \cup \alpha_2(V_1'))^c} = \text{id}.$$

Put $U_1 = \gamma^{-1}(V_1')$ so that $U_1 \subset V_1$ and

$$\alpha_1(U_1) \subset Y_1, \quad \alpha_1|_{(U_1 \cup \alpha_1(U_1))^c} = \text{id}, \quad \alpha_2\gamma(U_1) \subset Y_2, \quad \alpha_2|_{(\gamma(U_1) \cup \alpha_2\gamma(U_1))^c} = \text{id}.$$

Since $[U_1 \cup \alpha_1(U_1) \cup \gamma(U_1) \cup \alpha_2\gamma(U_1)]^c \subset (U_1 \cup \gamma(U_1))^c$ and $U_1 \cap \gamma(U_1) = \emptyset$, by putting

$$\alpha = \begin{cases} \alpha_1 & \text{on } U_1, \\ \alpha_2 & \text{on } \gamma(U_1), \\ \text{id} & \text{elsewhere,} \end{cases}$$

we have

$$\alpha^2 = \text{id}, \quad \alpha(U_1) \subset Y_1, \quad \alpha(\gamma(U_1)) \subset Y_2, \quad \alpha|_{[U_1 \cup \alpha(U_1) \cup \gamma(U_1) \cup \alpha\gamma(U_1)]^c} = \text{id}.$$

By a similar manner to the above discussion, we have $\beta \in \Gamma_A$ with $\beta^2 = \text{id}$ satisfying (3.2) and (3.4). We define $\psi \in \Gamma_A$ by setting

$$\psi = \begin{cases} \alpha & \text{on } U_1 \cup \alpha(U_1) \cup \gamma(U_1) \cup \alpha\gamma(U_1), \\ \beta & \text{on } \eta\alpha(U_1) \cup \beta\eta\alpha(U_1) \cup \eta\alpha\gamma(U_1) \cup \beta\eta\alpha\gamma(U_1), \\ \text{id} & \text{elsewhere.} \end{cases}$$

We then have

$$\psi^{-1}\eta\psi(U_1) \subset U'' \subset U, \quad \psi^{-1}\eta\psi(\gamma(U_1)) \subset \gamma(U''') \subset \gamma(U)$$

so that

$$\gamma(\psi^{-1}\eta\psi(U_1)) \cap (\psi^{-1}\eta\psi)(\gamma(U_1)) \subset \gamma(U'') \cap \gamma(U''') = \emptyset.$$

□

By using Lemma 3.1 and Lemma 3.2 with Lemma 2.6, one may show the following proposition in a similar manner to the proof of [10, Proposition 3.13].

Proposition 3.3. *If O is a regular open set of X_A , then the pair $(\Gamma_O, \Gamma_{O^\perp})$ is a strong commuting pair.*

4. THE CLOPEN CONDITION

In this section, we will find algebraic conditions of the strong commuting pair $(\Gamma_O, \Gamma_{O^\perp})$ of subgroups of Γ_A for a clopen set $O \in CO(X_A)$, and prove Proposition 4.12. The proposition comes from Lemma 4.11, which is based on Lemma 4.6 and Lemma 4.8. There are similar statements to Lemma 4.6 and Lemma 4.8 in [10] as [10, Lemma 3.18] and [10, Lemma 3.19]. In their proofs of the latter lemmas, [10, Lemma 3.3] has been used. In our setting, we do not have a version of [10, Lemma 3.3], so that we must modify the given proofs in [10] of [10, Lemma 3.18] and [10, Lemma 3.19]. In particular, we must provide several lemmas to prove Lemma 4.6.

Lemma 4.1. *For a word $\nu = (\nu_1, \dots, \nu_n) \in B_n(X_A)$ with $n > 1$ and a nonempty open set $V \subset X_A$ such that U_ν does not contain V , there exists $\alpha \in \Gamma_A$ such that*

$$\alpha(U_\nu) \subset V, \quad \alpha^2 = \text{id}, \quad \alpha|_{(U_\nu \cup \alpha(U_\nu))^c} = \text{id}.$$

Proof. Take $\mu = \mu_1 \cdots \mu_k \in B_k(X_A)$ such that $k > n$ and $(\mu_1, \dots, \mu_n) \neq (\nu_1, \dots, \nu_n)$. As A is irreducible, there exists a word $(\xi_1, \dots, \xi_l) \in B_l(X_A)$ such that $\mu\xi\nu \in B_*(X_A)$. Hence we have

$$U_{\mu\xi\nu} \subset U_\mu \subset V, \quad U_{\mu\xi\nu} \cap U_\nu = \emptyset.$$

Define $\alpha \in \Gamma_A$ by setting for $x \in X_A$

$$\alpha(x) = \begin{cases} (\mu_1, \dots, \mu_k, \xi_1, \dots, \xi_l, \nu_1, \dots, \nu_n, x_{n+1}, x_{n+2}, \dots) \in U_{\mu\xi\nu}, \\ \text{if } x = (\nu_1, \dots, \nu_n, x_{n+1}, x_{n+2}, \dots) \in U_\nu, \\ (\nu_1, \dots, \nu_n, x_{n+1}, x_{n+2}, \dots) \in U_\nu, \\ \text{if } x = (\mu_1, \dots, \mu_k, \xi_1, \dots, \xi_l, \nu_1, \dots, \nu_n, x_{n+1}, x_{n+2}, \dots) \in U_{\mu\xi\nu}, \\ x \quad \text{otherwise.} \end{cases}$$

Then α defines an element of Γ_A which has the desired properties. \square

Lemma 4.2. *For $U, V \in CO(X_A)$ with $V \setminus U \neq \emptyset$, there exist a clopen partition $U_1, \dots, U_n \subset U$ of U such that $\cup_{i=1}^n U_i = U$ and $U_i \cap U_j = \emptyset$ for $i \neq j$ and homeomorphisms $\alpha_1, \dots, \alpha_n \in \Gamma_A$ such that*

$$\alpha_i(U_i) \subset V \setminus U, \quad \alpha_i(U_i) \cap \alpha_j(U_j) = \emptyset, \quad \alpha_i^2 = \text{id}, \quad \alpha_i|_{(U_i \cup \alpha(U_i))^c} = \text{id}$$

for $i, j = 1, \dots, n$ with $i \neq j$.

Proof. Since U is clopen, there exist words $\nu(1), \dots, \nu(n) \in B_*(X_A)$ such that U is a disjoint union of cylinder sets $U = U_{\nu(1)} \sqcup \cdots \sqcup U_{\nu(n)}$. Put $\tilde{V} = V \setminus U$ a nonempty clopen set. Take $V_1 \subset \tilde{V}$ a clopen subset of \tilde{V} such that $V_1 \neq \tilde{V}$. For $U_{\nu(1)}$ and V_1 , Lemma 4.1 ensures that there exists $\alpha_1 \in \Gamma_A$ such that

$$\alpha_1(U_{\nu(1)}) \subset V_1, \quad \alpha_1^2 = \text{id}, \quad \alpha_1|_{(U_{\nu(1)} \cup \alpha_1(U_{\nu(1)}))^c} = \text{id}.$$

By applying Lemma 4.1, recursively, we have clopen sets $V_1, \dots, V_n \subset \tilde{V}$ with $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\alpha_i \in \Gamma_A, i = 1, \dots, n$ such that

$$\alpha_i(U_{\nu(i)}) \subset V_i, \quad \alpha_i^2 = \text{id}, \quad \alpha_i|_{(U_{\nu(i)} \cup \alpha_i(U_{\nu(i)}))^c} = \text{id}$$

for $i = 1, \dots, n$. By putting $U_i = U_{\nu(i)}$, the proof ends. \square

Lemma 4.3. *Let $U, W \in CO(X_A)$ be nonempty clopen sets such that $U \cap W = \emptyset$. Then there exists $\alpha \in \Gamma_A$ such that*

$$\alpha(U) \subset W, \quad \alpha^2 = \text{id}, \quad \alpha|_{(U \cup \alpha(U))^c} = \text{id}.$$

Proof. By the preceding lemma, we have clopen partitions: $U_1 \sqcup \cdots \sqcup U_n = U$, $W_1 \sqcup \cdots \sqcup W_n = W$ and homeomorphisms $\alpha_i \in \Gamma_A, i = 1, \dots, n$ such that

$$\alpha_i(U_i) \subset W_i, \quad \alpha_i^2 = \text{id}, \quad \alpha_i|_{(U_i \cup \alpha_i(U_i))^c} = \text{id}$$

for $i = 1, \dots, n$. The homeomorphisms $\alpha_i, i = 1, \dots, n$ commute with each other. The homeomorphism $\alpha = \alpha_1 \circ \cdots \circ \alpha_n \in \Gamma_A$ has the desired properties. \square

Lemma 4.4. *Let $O \in CO(X_A)$ be a clopen set. Let $U \subset O$, $V \subset O^c$ be Γ_A -equivalent nonempty clopen sets and $W \subset O$, $W' \subset O^c$ be nonempty clopen sets such that $U \cap W = \emptyset$, $V \cap W' = \emptyset$. Then there exist a clopen partition $U_1, \dots, U_n \subset U$ of U and a clopen partition $V_1, \dots, V_n \subset V$ of V such that $U = \sqcup_{i=1}^n U_i$, $V = \sqcup_{i=1}^n V_i$, $U_i \sim_{\Gamma_A} V_i$ for $i = 1, \dots, n$ and homeomorphisms $\alpha_i \in \Gamma_O$, $\beta_i \in \Gamma_{O^\perp}$ for $i = 1, \dots, n$ such that*

$$\alpha_i(U_i) \subset W, \quad \alpha_i(U_i) \cap \alpha_j(U_j) = \emptyset, \quad \alpha_i^2 = \text{id}, \quad \alpha_i|_{(U_i \cup \alpha(U_i))^c} = \text{id}, \quad (4.1)$$

$$\beta_i(V_i) \subset W', \quad \beta_i(V_i) \cap \beta_j(V_j) = \emptyset, \quad \beta_i^2 = \text{id}, \quad \beta_i|_{(V_i \cup \beta(V_i))^c} = \text{id} \quad (4.2)$$

for $i, j = 1, \dots, n$ with $i \neq j$.

Proof. Since $U \sim_{\Gamma_A} V$, there exists $\gamma \in \Gamma_A$ such that $\gamma(U) = V$. As $U \cap V = \emptyset$, by Lemma 2.2, one may assume that $\gamma^2 = \text{id}$ and $\gamma|_{(U \cup V)^c} = \text{id}$. Since γ is a cylinder map, there exist words $\nu(1), \dots, \nu(n), \mu(1), \dots, \mu(n) \in B_*(X_A)$ such that

$$U = \sqcup_{i=1}^n U_{\nu(i)}, \quad V = \sqcup_{i=1}^n U_{\mu(i)}, \quad \gamma(U_{\nu(i)}) = U_{\mu(i)}, \quad i = 1, \dots, n, \\ U_{\nu(i)} \cap U_{\nu(j)} = \emptyset, \quad U_{\mu(i)} \cap U_{\mu(j)} = \emptyset, \quad i \neq j.$$

Hence $U_{\nu(i)} \sim_{\Gamma_A} V_{\mu(i)}$, $i = 1, \dots, n$. By the proof of Lemma 4.2, one may find $\alpha_i \in \Gamma_A$ such that

$$\alpha_i(U_{\nu(i)}) \subset W, \quad \alpha_i(U_{\nu(i)}) \cap \alpha_j(U_{\nu(j)}) = \emptyset, \\ \alpha_i^2 = \text{id}, \quad \alpha_i|_{(U_{\nu(i)} \cup \alpha(U_{\nu(i)}))^c} = \text{id}$$

for $i, j = 1, \dots, n$ with $i \neq j$. As $U_{\nu(i)} \cup \alpha_i(U_{\nu(i)}) \subset O$, one sees that $\alpha_i \in \Gamma_O$. One may similarly find $\beta_i \in \Gamma_{O^\perp}$ having the desired properties by putting $U_i = U_{\nu(i)}$, $V_i = U_{\mu(i)}$ for $i = 1, \dots, n$. \square

For subsets H_1, H_2, H_3 of Γ_A , let us denote by $\langle H_1, H_2 \rangle$ and $\langle H_1, H_2, H_3 \rangle$ the subgroup of Γ_A generated by elements of H_1, H_2 and that of Γ_A generated by elements of H_1, H_2, H_3 respectively.

One may see a similar statement to the following two lemmas in [10, Lemma 3.18]. The proof of [10, Lemma 3.18] needs [10, Lemma 3.3] for which we do not have a corresponding lemma in our setting. The following proofs are different from the proof of [10, Lemma 3.18].

Lemma 4.5. *Let $O \in CO(X_A)$ and $\eta \in \Gamma_A$ satisfy $\eta(O) \cap O^c \neq \emptyset$, $\eta(O^c) \cap O^c \neq \emptyset$. Let $U \subset O$, $V \subset O^c$ be Γ_A -equivalent nonempty clopen sets such that $O \cap \eta^{-1}(O^c) \cap U^c \neq \emptyset$, $O^c \cap \eta^{-1}(O^c) \cap V^c \neq \emptyset$. Then there exists $\chi \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$ such that*

$$\chi(U) = V, \quad \chi(V) = U, \quad \chi|_{(U \cup V)^c} = \text{id}.$$

Proof. Put the nonempty clopen sets

$$W = O \cap \eta^{-1}(O^c) \cap U^c, \quad W' = O^c \cap \eta^{-1}(O^c) \cap V^c.$$

By the preceding lemma, there exist disjoint clopen partitions of U and of V :

$$U = U_1 \sqcup \dots \sqcup U_n, \quad V = V_1 \sqcup \dots \sqcup V_n$$

such that $U_i \underset{\Gamma_A}{\sim} V_i$ and $\alpha_i \in \Gamma_O$, $\beta_i \in \Gamma_{O^\perp}$ satisfying (4.1) and (4.2) for $i = 1, \dots, n$.

Put $U^i = \alpha_i(U_i) \subset W$, $V^i = \beta_i(V_i) \subset W'$. One then sees

$$\eta(U^i), \eta(V^i) \subset O^c, \quad \eta(U^i) \underset{\Gamma_A}{\sim} \eta(V^i), \quad \eta(U^i) \cap \eta(V^i) = \emptyset, \quad i = 1, \dots, n.$$

By Lemma 2.2, there exist $\gamma^i \in \Gamma_{O^\perp}$, $i = 1, \dots, n$ such that

$$\gamma^i(\eta(U^i)) = \eta(V^i), \quad (\gamma^i)^2 = \text{id}, \quad \gamma^i|_{(\eta(U^i) \cup \eta(V^i))^c} = \text{id}.$$

Then $\chi^i = \eta^{-1}\gamma^i\eta \in \langle \Gamma_{O^\perp}, \eta \rangle$ satisfies

$$\chi^i(U^i) = V^i, \quad \chi^i(V^i) = U^i, \quad i = 1, \dots, n.$$

For $x \in (U^i \cup V^i)^c$, we have $\gamma^i(\eta(x)) = \eta(x)$ and hence $\chi^i|_{(U^i \cup V^i)^c} = \text{id}$. We put

$$\chi_i = \alpha_i \beta_i \chi^i \alpha_i \beta_i \in \Gamma_O \Gamma_{O^\perp} \langle \Gamma_{O^\perp}, \eta \rangle \Gamma_O \Gamma_{O^\perp}, \quad i = 1, \dots, n.$$

It then follows that

$$\begin{aligned} \chi_i(U_i) &= \alpha_i \beta_i \chi^i \alpha_i(U_i) = \alpha_i \beta_i \chi^i(U^i) = \alpha_i \beta_i(V^i) = \alpha_i(V_i) = V_i, \\ \chi_i(V_i) &= \alpha_i \beta_i \chi^i \alpha_i(V_i) = \alpha_i \beta_i \chi^i(V^i) = \alpha_i \beta_i(U^i) = \alpha_i(U_i) = U_i, \\ (U_i \cup V_i)^c &= (\alpha_i(U^i) \cup \beta_i(V^i))^c = (\alpha_i \beta_i(U^i) \cup \alpha_i \beta_i(V^i))^c = \alpha_i \beta_i((U^i \cup V^i)^c). \end{aligned}$$

As $\alpha_i \beta_i \chi^i \alpha_i \beta_i|_{\alpha_i \beta_i((U^i \cup V^i)^c)} = \text{id}$, one sees $\chi_i|_{(U_i \cup V_i)^c} = \text{id}$. We set

$$\chi = \chi_1 \chi_2 \cdots \chi_n \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle.$$

Since $\chi_i \chi_j = \chi_j \chi_i$ for $i, j = 1, \dots, n$, we have

$$\chi(U) = V, \quad \chi(V) = U, \quad \chi|_{(U \cup V)^c} = \text{id}.$$

□

Lemma 4.6. *Let $O \in CO(X_A)$ and $\eta \in \Gamma_A$ satisfy $\eta(O) \cap O^c \neq \emptyset, \eta(O^c) \cap O \neq \emptyset$. Let $U \subset O$, $V \subset O^c$ be Γ_A -equivalent nonempty clopen sets. Then there exists $\chi \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$ such that*

$$\chi(U) = V, \quad \chi(V) = U, \quad \chi|_{(U \cup V)^c} = \text{id}.$$

Proof. We first assume that $U \neq O$ and $V \neq O^c$. If both of the conditions $O \cap \eta^{-1}(O^c) \cap U^c \neq \emptyset$ and $O^c \cap \eta^{-1}(O) \cap V^c \neq \emptyset$ hold, then the assertion follows from the previous lemma. We next assume that

$$O \cap \eta^{-1}(O^c) \cap U^c = \emptyset, \quad O^c \cap \eta^{-1}(O) \cap V^c = \emptyset$$

and hence

$$O \cap \eta^{-1}(O^c) \subset U, \quad O^c \cap \eta^{-1}(O) \subset V.$$

By Lemma 4.3, there exist clopen sets $U' \subset O$ and $V' \subset O^c$, and homeomorphisms $\alpha \in \Gamma_O$ and $\beta \in \Gamma_{O^\perp}$ such that

$$U' \cap U = \emptyset, \quad \alpha(U) = U', \quad \alpha^2 = \text{id}, \quad \alpha|_{(U \cup U')^c} = \text{id}, \quad (4.3)$$

and

$$V' \cap V = \emptyset, \quad \beta(V) = V', \quad \beta^2 = \text{id}, \quad \beta|_{(V \cup V')^c} = \text{id}. \quad (4.4)$$

Since U' and V' are Γ_A -equivalent, they satisfy the assumption of the preceding lemma, so that there exists $\tilde{\chi} \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$ such that

$$\tilde{\chi}(U') = V', \quad \tilde{\chi}(V') = U', \quad \tilde{\chi}|_{(U' \cup V')^c} = \text{id}.$$

We then see that the homeomorphism $\chi = \alpha \circ \beta \circ \tilde{\chi} \circ \alpha \circ \beta$ belongs to $\langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$. As $\beta(U) = U, \beta(V') = V, \alpha(V) = V$, it then follows that

$$\chi(U) = \alpha \circ \beta \circ \tilde{\chi} \circ \alpha(U) = \alpha \circ \beta \circ \tilde{\chi}(U') = \alpha \circ \beta(V') = \alpha(V) = V,$$

and similarly $\chi(V) = U$. We note that α commutes with β . For $x \in (U \cup V)^c$, we have $\alpha(x) \in (U')^c, \beta(x) \in (V')^c$ and

$$\alpha\beta(x) \in \alpha((V')^c) = (\alpha(V'))^c = (V')^c, \quad \beta\alpha(x) \in \beta((U')^c) = (\beta(U'))^c = (U')^c$$

so that $\alpha\beta(x) \in (U' \cup V')^c$. As $\tilde{\chi}|_{(U' \cup V')^c} = \text{id}$, one then sees

$$\chi(x) = \alpha\beta\tilde{\chi}\alpha\beta(x) = \alpha\beta\alpha\beta(x) = x.$$

This shows that $\chi|_{(U \cup V)^c} = \text{id}$.

If $O \cap \eta^{-1}(O^c) \cap U^c = \emptyset$ and $O^c \cap \eta^{-1}(O^c) \cap V^c \neq \emptyset$, then we may take $U' \subset O$ instead of U such that (4.3), and apply the preceding lemma for $U' \subset O$ and $V \subset O$. We then have $\chi' \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$ such that

$$\chi'(U') = V, \quad \chi'(V) = U', \quad \chi'|_{(U' \cup V)^c} = \text{id}.$$

By putting $\chi = \alpha \circ \chi' \circ \alpha$, we have a desired homeomorphism.

If $O \cap \eta^{-1}(O^c) \cap U^c \neq \emptyset$ and $O^c \cap \eta^{-1}(O^c) \cap V^c = \emptyset$, we symmetrically have a desired homeomorphism.

We will finally consider general clopen sets $U \subset O$ and $V \subset O^c$. The conditions $U \neq O, V \neq O^c$ are not necessarily assumed. Suppose that U and V are Γ_A -equivalent so that there exists $\gamma \in \Gamma_A$ such that $\gamma(U) = V$. Take nonempty clopen sets U^1, U^2 such that $U = U^1 \cup U^2$ and $U^1 \cap U^2 = \emptyset$. Put $V^1 = \gamma(U^1), V^2 = \gamma(U^2)$. Hence $U_i \sim_{\Gamma_A} V_i$. Since $U_i \neq O, V_i \neq O^c$, by the above discussions, one may find $\chi_i \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$ such that

$$\chi_i(U_i) = V_i, \quad \chi_i(V_i) = U_i, \quad \chi_i|_{(U_i \cup V_i)^c} = \text{id}, \quad i = 1, 2.$$

As $\chi_1\chi_2 = \chi_2\chi_1$, by setting $\chi = \chi_1 \circ \chi_2 \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$, we have

$$\chi(U) = V, \quad \chi(V) = U, \quad \chi|_{(U \cup V)^c} = \text{id}.$$

□

Lemma 4.7. *Let $O \in CL(X_A), \gamma \in \Gamma_A$ satisfy $\gamma(O) = O$, then there exist $\gamma_1 \in \Gamma_O, \gamma_2 \in \Gamma_{O^\perp}$ such that $\gamma = \gamma_1\gamma_2$ so that $\gamma \in \Gamma_O\Gamma_{O^\perp}$.*

Proof. Assume that $\gamma(O) = O$. We set for $x \in \Gamma_A$

$$\gamma_1(x) = \begin{cases} \gamma(x) & \text{if } x \in O, \\ x & \text{if } x \in O^c, \end{cases}, \quad \gamma_2(x) = \begin{cases} x & \text{if } x \in O, \\ \gamma(x) & \text{if } x \in O^c. \end{cases}$$

The homeomorphisms γ_1, γ_2 satisfy $\gamma_1 \in \Gamma_O, \gamma_2 \in \Gamma_{O^\perp}$ and $\gamma = \gamma_1\gamma_2$. □

The same statement as the following lemma is seen in [10, Lemma 3.19]. The proof of [10, Lemma 3.19] however does not well work in our setting. We give a different proof as in the following way.

Lemma 4.8. *Let $O \in CL(X_A)$ and $\eta \in \Gamma_A$ satisfy $\eta(O) \cap O^c \neq \emptyset, \eta(O^c) \cap O^c \neq \emptyset$. Then the subgroup $\langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$ coincides with Γ_A .*

Proof. For an arbitrary fixed homeomorphism $\psi \in \Gamma_A$, we will show that $\psi \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$. By Lemma 4.3, there exist $\alpha_1 \in \Gamma_A$ such that $\alpha_1(O) \subset O^c, \alpha_1^2 = \text{id}, \alpha_1|_{(O \cup \alpha(O))^c} = \text{id}$. Put $V_1 = \alpha_1(O)$. By Lemma 4.6, there exists $\chi_1 \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$ such that $\chi_1(O) = V_1, \chi_1(V_1) = O, \chi_1|_{(O \cup V_1)^c} = \text{id}$. We have two cases.

Case 1: $O \cap \psi^{-1}(O^c) \neq \emptyset$.

Since $O \cap \psi^{-1}(O^c) \subset O$ and $V_1 \subset O^c$, by Lemma 4.3, there exists $\alpha_2 \in \Gamma_A$ such that $\alpha_2(V_1) \subset O \cap \psi^{-1}(O^c)$. Put $V_2 = \alpha_2(V_1)$. By Lemma 4.6, one may take $\chi_2 \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$ such that $\chi_2(V_1) = V_2$. Put $W_2 = \psi(V_2) \subset O^c$ so that $\psi\chi_2\chi_1(O) = W_2 \subset O^c$. Since $O \sim_{\Gamma_A} W_2$, by Lemma 4.6, we have $\chi_3 \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$ such that $\chi_3(O) = W_2$. Thus we have $\chi_3(O) = \psi\chi_2\chi_1(O)$ so that by Lemma 4.7, $\chi_3^{-1}\psi\chi_2\chi_1 \in \Gamma_O\Gamma_{O^\perp}$. Since $\chi_1, \chi_2, \chi_3 \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$, we conclude that $\psi \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$.

Case 2: $O \cap \psi^{-1}(O^c) = \emptyset$.

Since $\psi(O) \cap O^c = \emptyset$, there exists $\alpha \in \Gamma_A$ such that $\alpha(\psi(O)) \subset O^c$. Put $V = \alpha(\psi(O))$. By Lemma 4.6, there exists $\chi \in \langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$ such that $\chi(\psi(O)) = V$. Put $\tilde{\psi} = \chi \circ \psi$. As $\tilde{\psi}(O) \subset O^c$, by using Case 1 for $\tilde{\psi}$ we see that $\tilde{\psi}$ and hence ψ belongs to $\langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle$. \square

The following lemma is seen in [10, Lemma 3.20].

Lemma 4.9 ([10, Lemma 3.20]). *For a regular open set $O \in RO(X_A)$, the following conditions are equivalent:*

- (i) O is clopen.
- (ii) For all $U \in RO(X_A)$ with $O \subset U$ and $O \neq U$, we have $\overline{O} \subset U$.

Lemma 4.10. *Let $E \subset X_A$ be a nonempty closed of X_A such that $\gamma(E) \subset E$ for all $\gamma \in \Gamma_A$. Then we have $E = X_A$. This means that the action of Γ_A on X_A is minimal.*

Proof. Suppose that $E \neq X_A$. Take an arbitrary point $x \in E$. Since E^c is a nonempty open set, one may find clopen sets $U, V \subset X_A$ such that

$$x \in U, \quad U \cap V = \emptyset, \quad V \subset E^c.$$

By Lemma 4.3, there exists $\gamma \in \Gamma_A$ such that $\gamma(U) \subset V$. Since $\gamma(x) \in V \subset E^c$, we have a contradiction, so that $E = X_A$. \square

A similar statement to the following lemma is seen in [10, Lemma 3.21]. The proof below is also similar to that of [10, Lemma 3.21]. We give the proof for the sake of completeness.

Lemma 4.11 (cf. [10, Lemma 3.21]). *Let $O \subset X_A$ be a regular open set. Then the following two conditions are equivalent:*

- (i) O is clopen.
- (ii) For a strong commuting pair (H, K) of subgroups of Γ_A such that $\Gamma_O \subset H$ with $\Gamma_O \neq H$, the subgroup $\langle H, \Gamma_{O^\perp} \rangle$ generated by H and Γ_{O^\perp} coincides with Γ_A .

Proof. (i) \implies (ii): Assume that O is clopen. Let (H, K) be a strong commuting pair of subgroups of Γ_A satisfying $\Gamma_O \subset H$ with $\Gamma_A \neq H$. Suppose that there exists $\eta \in H$ such that $\eta(O^c) \subset O$. Then we have $\eta^{-1}\Gamma_{O^\perp}\eta \subset \Gamma_O$ so that $\Gamma_{O^\perp} \subset \eta\Gamma_O\eta^{-1} \subset H$. Therefore by Lemma 2.7, we have

$$K = H^\perp \subset (\Gamma_{O^\perp})^\perp = \Gamma_O \subset H.$$

As (H, K) is a strong commuting pair, we have $K = \{\text{id}\}$ and hence $H = \Gamma_A$ so that $\langle H, \Gamma_{O^\perp} \rangle = \Gamma_A$. Suppose next that $\eta(O^c) \cap O^c \neq \emptyset$ for all $\eta \in H$. If $\eta(O) \subset O$ for all $\eta \in H$, then $\eta(O) = O$. Then Γ_O is a normal subgroup of H so that $(\Gamma_{O^\perp})^\perp = K$ and hence $H = \Gamma_O$. Therefore we may assume that there exists $\eta \in H$ such that

$$\eta(O^c) \cap O^c \neq \emptyset, \quad \eta(O) \cap O^c \neq \emptyset.$$

Hence by Lemma 4.8, we have $\langle \Gamma_O, \Gamma_{O^\perp}, \eta \rangle = \Gamma_A$ so that $\langle H, \Gamma_{O^\perp} \rangle = \Gamma_A$.

(ii) \implies (i): We will show that if $U \in RO(X_A)$ with $O \subset U$ and $O \neq U$, then $\overline{O} \subset U$. For $U \in RO(X_A)$, consider the pair $(\Gamma_U, \Gamma_{U^\perp})$ of subgroups of Γ_A . As $O \subset U$ with $O \neq U$, one has $\Gamma_O \subset \Gamma_U$ with $\Gamma_O \neq \Gamma_U$ by Lemma 2.7 (i) so that $\langle \Gamma_U, \Gamma_{O^\perp} \rangle = \Gamma_A$ by the condition (ii). Assume that $\overline{O} \cap U^c \neq \emptyset$. Since \overline{O} is fixed by Γ_{O^\perp} and U^c is fixed by Γ_U , the closed set $\overline{O} \cap U^c$ is fixed by $\langle \Gamma_U, \Gamma_{O^\perp} \rangle$. Hence by the preceding lemma, $\overline{O} \cap U^c = X_A$ a contradiction. Therefore we have $\overline{O} \subset U$ and hence O is clopen. \square

Following [10], we define condition (D3) for a strong commuting pair (H, K) as follows:

Definition. A strong commuting pair (H, K) of subgroups of Γ_A is said to satisfy condition (D3) if it satisfies the following two conditions (a) and (b):

- (a) For a strong commuting pair (H', K') of subgroups of Γ_A such that $H \subset H'$ with $H \neq H'$, the subgroup $\langle H', K' \rangle$ of Γ_A generated by H' and K' is equal to Γ_A .
- (b) For a strong commuting pair (H'', K'') of subgroups of Γ_A such that $K \subset K''$ with $K \neq K''$, the subgroup $\langle H, K'' \rangle$ of Γ_A generated by H and K'' is equal to Γ_A .

We remark that in the statement of the above condition (b), the condition $K \subset K''$ with $K \neq K''$ is equivalent to the condition $H'' \subset H$ with $H'' \neq H$. Therefore we have

Proposition 4.12. *Let $O \subset X_A$ be a regular open set. Then O is clopen if and only if the strong commuting pair $(\Gamma_O, \Gamma_{O^\perp})$ satisfies condition (D3).*

Proof. A regular open set $O \subset X_A$ is clopen if and only if O^\perp is clopen. Hence the assertion is clear by Lemma 4.11. \square

5. SUPPORT OF A STRONG COMMUTING PAIR (H, K)

As in the preceding section, a clopen set O (and O^\perp) yields a strong commuting pair $(\Gamma_O, \Gamma_{O^\perp})$ of subgroups of Γ_A satisfying condition (D3) (Proposition 4.12). In this section, we will conversely define a clopen set P_H (and P_K) of X_A from a strong commuting pair (H, K) satisfying condition (D3).

Following [10, Definition 3.14], we use the notations below.

For $\gamma \in \Gamma_A$, denote by X_A^γ the set of elements of X_A fixed by γ :

$$X_A^\gamma = \{x \in X_A \mid \gamma(x) = x\}.$$

Denote by P_γ the support of γ defined by

$$P_\gamma = \overline{(X_A^\gamma)^c}.$$

We note that P_γ is a regular closed set and hence

$$P_\gamma = \overline{(P_\gamma)^\circ} = (X_A^\gamma)^\perp.$$

We in fact see that the inclusion relation $\overline{(P_\gamma)^\circ} \subset P_\gamma$ is clear. For the converse inclusion relation, we see $\overline{((X_A^\gamma)^c)^\circ} \supset ((X_A^\gamma)^c)^\circ = (X_A^\gamma)^c$. Hence $\overline{(P_\gamma)^\circ} \supset P_\gamma$.

For a subset $H \subset \text{Homeo}(X_A)$, define the support of H as a closed subset of X_A by

$$P_H = \overline{\cup_{\eta \in H} (P_\eta)^\circ}.$$

Remark.

1. P_H is a regular closed set such that both P_H and $(P_H)^\circ$ are H^\perp -invariant.

We in fact see that the inclusion relation $\overline{(P_H)^\circ} \subset P_H$ is clear. Conversely, the inclusion relation $\cup_{\eta \in H} (P_\eta)^\circ \subset P_H$ implies $\cup_{\eta \in H} (P_\eta)^\circ \subset (P_H)^\circ$ so that $P_H \subset \overline{(P_H)^\circ}$.

We will next see that $\xi(P_H) = P_H$ for $\xi \in H^\perp$. For $x \in X_A^\eta$ with $\eta \in H$, we have $\eta(\xi(x)) = \xi(\eta(x)) = \xi(x)$ so that $\xi(X_A^\eta) \subset X_A^\eta$ and similarly $\xi^{-1}(X_A^\eta) \subset X_A^\eta$. Hence we have $\xi(X_A^\eta) = X_A^\eta$. This implies $\xi(P_\eta) = P_\eta$ for $\eta \in H$. As ξ is a homeomorphism, we have $\xi((P_\eta)^\circ) = (P_\eta)^\circ$. Hence we have $\xi(P_H) = P_H$ for $\xi \in H^\perp$ and $\xi((P_H)^\circ) = (P_H)^\circ$ for $\xi \in H^\perp$.

2. For $\gamma \in \Gamma_A$, the set P_γ is clopen.

Its proof is the following. As γ is a cylinder map, there exist $L \in \mathbb{N}$ and words $\mu(\nu) = (\mu_1(\nu), \dots, \mu_{k(\nu)}(\nu)) \in B_{k(\nu)}(X_A)$ for $\nu = (\nu_1, \dots, \nu_L) \in B_L(X_A)$ such that

$$\gamma(\nu_1, \dots, \nu_L, x_{L+1}, x_{L+2}, x_{L+3}, \dots) = (\mu_1(\nu), \dots, \mu_{k(\nu)}(\nu), x_{L+1}, x_{L+2}, x_{L+3}, \dots)$$

for $(\nu_1, \dots, \nu_L, x_{L+1}, x_{L+2}, x_{L+3}, \dots) \in U_\nu$. Hence the set $(X_A^\gamma)^c$ is a disjoint union of the following two clopen sets:

$$\begin{aligned} & \sqcup_{\nu \in B_L(X_A)} \{U_\nu \mid L \neq k(\nu)\}, \\ & \sqcup_{\nu \in B_L(X_A)} \{U_\nu \mid L = k(\nu), (\nu_1, \dots, \nu_L) \neq (\mu_1(\nu), \dots, \mu_{k(\nu)}(\nu))\}. \end{aligned}$$

This implies that $(X_A^\gamma)^c$ is clopen, and so is P_γ .

3. For $H \subset \text{Homeo}(X_A)$ and $U \subset RO(X_A)$, if $H \subset \Gamma_U$, then $P_H \subset \overline{U}$.

The proof is the following. In general, $H_1 \subset H_2 \subset \text{Homeo}(X_A)$ implies $P_{H_1} \subset P_{H_2}$. Hence the condition $H \subset \Gamma_U$ implies $P_H \subset P_{\Gamma_U}$. By the lemma below, one has $P_{\Gamma_U} = \overline{U}$ so that $P_H \subset \overline{U}$.

Lemma 5.1 (cf. [10, Lemma 3.16]). *If $O \subset X_A$ is an open set, we have $P_{\Gamma_O} = \overline{O}$. Hence for a clopen set $O \subset X_A$, we have $P_{\Gamma_O} = O$.*

Proof. For $\eta \in \Gamma_O$ and $x \in (X_A^\eta)^c$, one sees that $\eta(x) \neq x$ so that $x \in O$. Hence we have $(X_A^\eta)^c \subset O$, so that $P_\eta \subset \overline{O}$ and hence we have $P_{\Gamma_O} \subset \overline{O}$. For the converse inclusion relation, take $x \in O$. By Lemma 2.1, there exist an open neighborhood U of x and $\gamma \in \Gamma_U$ such that $x \in U \subset O$ and $\gamma(x) \neq x$. Take a clopen set V such that $x \in V \subset U$ and $\gamma(V) \cap V = \emptyset$. We thus have $V \subset (X_A^\gamma)^c$ so that $V \subset \overline{(X_A^\gamma)^c} = P_\gamma$ and $V \subset (P_\gamma)^\circ$. Hence $x \in \cup_{\gamma \in \Gamma_U} (P_\gamma)^\circ$. Since $\Gamma_U \subset \Gamma_O$, we have $x \in \cup_{\gamma \in \Gamma_O} (P_\gamma)^\circ = P_{\Gamma_O}$. Therefore we have $O \subset P_{\Gamma_O}$ and hence $\overline{O} \subset P_{\Gamma_O}$. \square

We will next show that if (H, K) satisfies condition (D3), then the sets P_H and P_K are both clopen. We provide a lemma.

Lemma 5.2. *Let H be a subgroup of Γ_A .*

- (i) $\eta(y) = y$ for all $\eta \in H$ and $y \in (P_H)^c$.
- (ii) Put $O = (P_H)^\circ$. Then we have $\zeta(O) = O$ and $\zeta(O^c) = O^c$ for all $\zeta \in H^\perp$.

Proof. (i) Since we have

$$(P_H)^c = \overline{(\cup_{\eta \in H} (P_\eta)^\circ)^c} = (\cap_{\eta \in H} ((P_\eta)^\circ)^c)^\circ \subset \cap_{\eta \in H} ((P_\eta)^\circ)^c$$

and $((P_\eta)^\circ)^c = \overline{(X_A^\eta)^\circ} \subset X_A^\eta$, we have $(P_H)^c \subset \cap_{\eta \in H} X_A^\eta$.

(ii) We note that P_η is clopen for $\eta \in H$. We have for $\zeta \in H^\perp$, $\zeta(P_H) = \cup_{\eta \in H} \zeta(X_A^\eta)^c$. Since ζ commutes with $\eta \in H$, we have $\zeta(X_A^\eta) = X_A^\eta$. Hence we see that $\zeta(P_H) = P_H$ so that $\zeta(O) = O$ and $\zeta(O^c) = O^c$. \square

Lemma 5.3 (cf. [10, Lemma 3.23]). *Let (H, K) be a strong commuting pair of subgroups of Γ_A satisfying condition (D3). Then the sets P_H and P_K are both clopen.*

Proof. Put $O = (P_H)^\circ$. As P_H is a regular closed set, O is a regular open set satisfying $\overline{O} = P_H$. We have a strong commuting pair $(\Gamma_O, \Gamma_{O^\perp})$ of subgroups of Γ_A . We will prove that $P_H = O$. By Lemma 5.1, one knows that $\overline{O} = P_{\Gamma_O}$ so that $P_H = P_{\Gamma_O}$. By the above lemma, $\eta(y) = y$ for all $\eta \in H$ and $y \in (P_H)^c$. As $O^c = \overline{(P_H)^c}$, we have $\eta(y) = y$ for all $\eta \in H$ and $y \in O^c$ so that $H \subset \Gamma_O$. We have two cases.

Case 1: $H = \Gamma_O$.

Since $K = H^\perp = (\Gamma_O)^\perp = \Gamma_{O^\perp}$, we have $(H, K) = (\Gamma_O, \Gamma_{O^\perp})$. By the hypothesis, we see that $(\Gamma_O, \Gamma_{O^\perp})$ satisfies condition (D3). Hence O is clopen by Proposition 4.12 so that $P_H = \overline{O} = O$ is clopen.

Case 2: $H \neq \Gamma_O$.

Suppose that $\overline{O} \cap O^c \neq \emptyset$. Since (H, K) satisfies condition (D3) and $H \subset \Gamma_O$ with $H \neq \Gamma_O$, we have $\langle \Gamma_O, K \rangle = \Gamma_A$. As $O = (P_H)^\circ$, we know $\zeta(O) = O$ for all $\zeta \in H^\perp$ so that $\zeta(\overline{O}) = \overline{O}$ for all $\zeta \in H^\perp = K$. Hence the closed set $\overline{O} \cap O^c$ is globally invariant under $\langle \Gamma_O, K \rangle = \Gamma_A$. By Lemma 4.10, one knows that $\overline{O} \cap O^c = \emptyset$ so that $\overline{O} = O$. Hence O is clopen such that $P_H = O$.

Symmetrically by considering $U = (P_K)^\circ$ one sees that P_K is clopen. \square

6. DYE PAIRS

This section is devoted to proving Proposition 6.6 which asserts that a strong commuting pair (H, K) with extra conditions (D4) and (D5) is of the form $(\Gamma_O, \Gamma_{O^\perp})$ for some clopen set O of X_A . The key lemma is Lemma 6.3 below. We use the same conditions (D4) and (D5) as [10, Definition 3.25].

Definition. A strong commuting pair (H, K) is said to satisfy conditions (D4) and (D5) if it satisfies the following conditions:

(D4) For a homeomorphism $\alpha \in \Gamma_A \setminus HK$, there exists $\eta \in H$ with $\eta \neq \text{id}$ (resp. $\kappa \in K$ with $\kappa \neq \text{id}$) such that $\alpha\eta\alpha^{-1} \in K$ (resp. $\alpha\kappa\alpha^{-1} \in H$).

(D5) If N is a subgroup of Γ_A with $N \neq \text{id}$ such that $\eta N \eta^{-1} = N$ for all $\eta \in H$, and $N \not\subset K$, (resp. $\kappa N \kappa^{-1} = N$ for all $\kappa \in K$, and $N \not\subset H$), then $N \cap H \neq \{\text{id}\}$ (resp. $N \cap K \neq \{\text{id}\}$).

Following [10], we will define the notion of Dye pair as in the following way.

Definition. A strong commuting pair (H, K) satisfying condition (D3) is said to be a *Dye pair* if it satisfies the conditions (D4) and (D5).

The following two lemmas and their proofs are similar to [10, Lemma 3.26] and [10, Lemma 3.27]. We omit their proofs.

Lemma 6.1 (cf. [10, Lemma 3.26]). *If O is a clopen set, then $(\Gamma_O, \Gamma_{O^\perp})$ is a Dye pair.*

Lemma 6.2 (cf. [10, Lemma 3.27]). *Let (H, K) be a strong commuting pair of subgroups of X_A satisfying conditions (D4) and (D5) such that $P_H = P_K = X_A$. If $O \subset X_A$ is either an H - or K -invariant nonempty open set of X_A , then $\overline{O} = X_A$.*

For a subgroup H of Γ_A , we define the continuous full group $[H]_c$ of H as follows. A homeomorphism γ on X_A belongs to $[H]_c$ if there exist a finite clopen partition $\sqcup_{i=1}^n U_i = X_A$ of X_A and $\eta_i \in H$ such that $X_A = \sqcup_{i=1}^n \eta_i(U_i)$ and $\gamma(x) = \eta_i(x)$ for $x \in U_i$ (see [10, Definition 2.2]). Following [10], for two homeomorphisms α, β on X_A , the closed set $F(\alpha, \beta)$ of X_A is defined by

$$F(\alpha, \beta) = \{x \in X_A \mid \alpha(x) = \beta(x)\}.$$

The statement of the following lemma is similar to [10, Lemma 3.24]. An invariant measure is used in the proof of [10, Lemma 3.24]. As in Proposition 2.4, there is no Γ_A -invariant regular Borel probability measure on X_A , so that we must modify its proof as in the following way.

Lemma 6.3 (cf. [10, Lemma 3.24]). *Let (H, K) be a strong commuting pair of subgroups of X_A satisfying conditions (D4) and (D5). Suppose that $P_H = P_K = X_A$. Then we have $[H]_c \cap [K]_c = \{\text{id}\}$.*

Proof. Suppose that $[H]_c \cap [K]_c \neq \{\text{id}\}$. Take $\alpha \in [H]_c \cap [K]_c$ with $\alpha \neq \text{id}$. There exist a nonempty clopen set $U_0 \in CO(X_A)$ and homeomorphisms $\eta_0 \in H, \kappa_0 \in K$ such that

$$\alpha(x) = \eta_0(x) = \kappa_0(x), \quad x \in U_0.$$

Hence we have

$$\emptyset \neq U_0 \subset F(\eta_0, \kappa_0)^\circ \subset F(\eta_0, \kappa_0) \neq X_A.$$

If $\eta_1, \eta_2 \in H$ satisfy $F(\eta_1, \eta_2)^\circ \neq \emptyset$, we have for $x \in F(\eta_1, \eta_2)^\circ$ and $\kappa \in K$

$$\eta_1(\kappa(x)) = \kappa(\eta_1(x)) = \kappa(\eta_2(x)) = \eta_2(\kappa(x)),$$

so that $F(\eta_1, \eta_2)^\circ$ is a K -invariant open set. By Lemma 6.2, we see that $F(\eta_1, \eta_2)^\circ$ is dense in X_A and hence $\eta_1 = \eta_2$ on X_A . Let $C(\eta_0)$ be the conjugacy class $\{\zeta\eta_0\zeta^{-1} \in H \mid \zeta \in H\}$ of η_0 in H . We similarly define the conjugacy class $C(\kappa_0)$ of κ_0 in K . For $\alpha, \beta \in C(\eta_0)$ with $\alpha \neq \beta$, we have $F(\alpha, \beta)^\circ = \emptyset$. For $\eta \in H$ and $\alpha \in C(\eta_0)$, we have

$$\eta\alpha\eta^{-1}(\eta(x)) = \eta\alpha(x) = \eta(\kappa_0(x)) = \kappa_0(\eta(x)), \quad x \in F(\alpha, \kappa_0)^\circ.$$

Hence $\eta(F(\alpha, \kappa_0)^\circ) \subset F(\eta\alpha\eta^{-1}, \kappa_0)^\circ$ and $\eta^{-1}(F(\eta\alpha\eta^{-1}, \kappa_0)^\circ) \subset F(\alpha, \kappa_0)^\circ$. We thus have with symmetric discussions for $C(\kappa_0)$,

- (i) $F(\alpha, \kappa_0)^\circ \cap F(\beta, \kappa_0)^\circ \subset F(\alpha, \beta)^\circ = \emptyset$ for $\alpha, \beta \in C(\eta_0)$ with $\alpha \neq \beta$,
 $F(\eta_0, \alpha)^\circ \cap F(\eta_0, \beta)^\circ \subset F(\alpha, \beta)^\circ = \emptyset$ for $\alpha, \beta \in C(\kappa_0)$ with $\alpha \neq \beta$.
- (ii) $\eta(F(\alpha, \kappa_0)^\circ) = F(\eta\alpha\eta^{-1}, \kappa_0)^\circ$ for $\eta \in H, \alpha \in C(\eta_0)$,
 $\kappa(F(\eta_0, \beta)^\circ) = F(\eta_0, \kappa\beta\kappa^{-1})^\circ$ for $\kappa \in K, \beta \in C(\kappa_0)$.

Let

$$N_H = \langle \Gamma_{F(\alpha, \kappa_0)^\circ} : \alpha \in C(\eta_0) \rangle \quad \text{and} \quad N_K = \langle \Gamma_{F(\eta_0, \beta)^\circ} : \beta \in C(\kappa_0) \rangle$$

be the subgroup of Γ_A generated by elements of $\cup_{\alpha \in C(\eta_0)} \Gamma_{F(\alpha, \kappa_0)^\circ}$ and that of Γ_A generated by elements of $\cup_{\beta \in C(\kappa_0)} \Gamma_{F(\eta_0, \beta)^\circ}$ respectively. Since for $\eta \in H, \alpha \in C(\eta_0)$

$$\eta\Gamma_{F(\alpha, \kappa_0)^\circ}\eta^{-1} = \Gamma_{\eta(F(\alpha, \kappa_0)^\circ)} = \Gamma_{F(\eta\alpha\eta^{-1}, \kappa_0)^\circ}, \quad (6.1)$$

we know that

$$\eta N_H \eta^{-1} = N_H \text{ for all } \eta \in H, \text{ and similarly } \kappa N_K \kappa^{-1} = N_K \text{ for all } \kappa \in K.$$

We will first show that $N_K \not\subset H$ and $N_H \not\subset K$. Suppose that $N_K \subset H$. We then have $\Gamma_{F(\eta_0, \beta)^\circ} \subset H$ for all $\beta \in C(\kappa_0)$, in particular, $\Gamma_{F(\eta_0, \kappa_0)^\circ} \subset H$. By (6.1), we see that $\Gamma_{F(\alpha, \kappa_0)^\circ} \subset H$ for $\alpha \in C(\eta_0)$. Hence we have $N_H \subset H$. Put for $x \in X_A$

$$\begin{aligned} \tilde{\eta}_0(x) &= \begin{cases} \eta_0(x) & \text{if } x \in F(\eta_0, \kappa_0)^\circ, \\ x & \text{otherwise,} \end{cases} \\ \tilde{\kappa}_0(x) &= \begin{cases} \kappa_0(x) & \text{if } x \in F(\eta_0, \kappa_0)^\circ, \\ x & \text{otherwise.} \end{cases} \end{aligned}$$

Since $\eta_0(F(\eta_0, \kappa_0)^\circ) = F(\eta_0, \kappa_0)^\circ$ and $\tilde{\eta}_0|_{(F(\eta_0, \kappa_0)^\circ)^c} = \text{id}$, one sees that $\tilde{\eta}_0 \in \Gamma_{F(\eta_0, \kappa_0)^\circ} \subset H$. We will next show $\tilde{\kappa}_0 \in K$. Take an arbitrary $\eta \in \Gamma_{F(\eta_0, \kappa_0)^\circ}$. For $x \in F(\eta_0, \kappa_0)^\circ$, we have

$$\eta \tilde{\kappa}_0(x) = \eta \kappa_0(x) = \kappa_0 \eta(x) = \tilde{\kappa}_0 \eta(x).$$

For $x \in (F(\eta_0, \kappa_0)^\circ)^c$, we have

$$\eta \tilde{\kappa}_0(x) = \eta(x) = \tilde{\kappa}_0 \eta(x).$$

Hence we have

$$\eta \tilde{\kappa}_0 = \tilde{\kappa}_0 \eta \text{ for all } \eta \in \Gamma_{F(\eta_0, \kappa_0)^\circ}. \quad (6.2)$$

Take an arbitrary $\eta \in \Gamma_{F(\alpha, \kappa_0)^\circ}$ with $\alpha \in C(\eta_0)$ such that $F(\alpha, \kappa_0)^\circ \cap F(\eta_0, \kappa_0)^\circ = \emptyset$. For $x \in F(\eta_0, \kappa_0)^\circ$, we have

$$\eta \tilde{\kappa}_0(x) = \eta \kappa_0(x) = \kappa_0 \eta(x) = \kappa_0(x) = \tilde{\kappa}_0(x) = \tilde{\kappa}_0 \eta(x).$$

For $x \in (F(\eta_0, \kappa_0)^\circ)^c$, we have

$$\eta \tilde{\kappa}_0(x) = \eta(x) = \tilde{\kappa}_0 \eta(x).$$

Hence we have

$$\eta \tilde{\kappa}_0 = \tilde{\kappa}_0 \eta \text{ for all } \eta \in \Gamma_{F(\alpha, \kappa_0)^\circ} \text{ with } F(\alpha, \kappa_0)^\circ \cap F(\eta_0, \kappa_0)^\circ = \emptyset. \quad (6.3)$$

By (6.2) and (6.3), we have $\tilde{\kappa}_0 \in (N_H)^\perp = K$ so that

$$\text{id} \neq \tilde{\eta}_0 = \tilde{\kappa}_0 \in H \cap K = \{\text{id}\}$$

a contradiction. Therefore we conclude that $N_K \not\subset H$ and similarly $N_H \not\subset K$.

Now $N_K \neq \{\text{id}\}$ such that $\kappa N_K \kappa^{-1} = N_K$ for all $\kappa \in K$ and $N_K \not\subset H$. By condition (D5), we see that $N_K \cap K \neq \{\text{id}\}$ and similarly $N_H \cap H \neq \{\text{id}\}$. We set

$$\tilde{N}_K = N_K \cap K, \quad \tilde{N}_H = N_H \cap H$$

so that \tilde{N}_K is a normal subgroup of K and \tilde{N}_H is a normal subgroup of H . By condition (D2), we see that

$$(\tilde{N}_K)^\perp = H, \quad (\tilde{N}_H)^\perp = K.$$

Let $\tilde{\eta}_0$ and $\tilde{\kappa}_0$ be previously defined homeomorphisms on X_A . In this setting we will show that $\tilde{\eta}_0 \in H$ and $\tilde{\kappa}_0 \in K$ as follows:

Since $F(\eta_0, \beta)^\circ, F(\eta_0, \beta')^\circ$ for $\beta, \beta' \in C(\kappa_0)$ are disjoint or equal, we have for any $\gamma \in \Gamma_{F(\eta_0, \alpha)^\circ}, \alpha \in C(\kappa_0)$

$$\gamma(F(\eta_0, \beta)^\circ) = F(\eta_0, \beta)^\circ \quad \text{for } \beta \in C(\kappa_0).$$

Hence each $F(\eta_0, \beta)^\circ$ for $\beta \in C(\kappa_0)$ is globally invariant under N_K and \tilde{N}_K . Let $\kappa \in \tilde{N}_K$ be an arbitrary element. For $x \in F(\eta_0, \kappa_0)^\circ$, we have

$$\kappa \tilde{\eta}_0(x) = \kappa \eta_0(x) = \eta_0 \kappa(x) = \tilde{\eta}_0 \kappa(x).$$

For $x \in (F(\eta_0, \kappa_0)^\circ)^c$, we have

$$\kappa \tilde{\eta}_0(x) = \kappa(x) = \tilde{\eta}_0 \kappa(x).$$

Hence we have $\kappa \tilde{\eta}_0 = \tilde{\eta}_0 \kappa$ for all $\kappa \in \tilde{N}_K$ so that we have $\tilde{\eta}_0 \in (\tilde{N}_K)^\perp = H$, symmetrically $\tilde{\kappa}_0 \in (\tilde{N}_H)^\perp = K$. Hence we have

$$\text{id} \neq \tilde{\eta}_0 = \tilde{\kappa}_0 \in H \cap K = \{\text{id}\}$$

a contradiction. Therefore we conclude that $[H]_c \cap [K]_c = \{\text{id}\}$. \square

Lemma 6.4. *Let (H, K) be a strong commuting pair of subgroups of X_A satisfying conditions (D4) and (D5). Suppose that $P_H = P_K = X_A$. Then there exists $\alpha \in [H]_c$ such that the fixed point set X_A^α is not K -invariant.*

Proof. As $P_H = X_A$, there exists $\gamma \in H$ and $x \in X_A$ such that $\gamma(x) \neq x$. One may take a clopen neighborhood V of x such that

$$\gamma(V) \cap V = \emptyset, \quad \overline{\gamma(V) \cup V} \neq X_A. \quad (6.4)$$

Define $\alpha \in [H]_c$ by setting for $x \in X_A$

$$\alpha(x) = \begin{cases} \gamma(x) & \text{for } x \in V, \\ \gamma^{-1}(x) & \text{for } x \in \gamma(V), \\ x & \text{for } x \in (V \cup \gamma(V))^c \end{cases}$$

so that $(X_A^\alpha)^c = \gamma(V) \cup V$. If X_A^α is K -invariant, $(X_A^\alpha)^c$ is a K -invariant nonempty open set of X_A . By Lemma 6.2, one sees that $(X_A^\alpha)^c$ is dense in X_A . This contradicts to (6.4). Hence X_A^α is not K -invariant. \square

Before reaching the final proposition, we provide a lemma below.

Lemma 6.5 (cf. [10, Lemma 3.17]). *Let (H, K) be a strong commuting pair of Γ_A .*

- (i) *For a nonempty clopen set $U \subset P_H$ such that $\gamma(U) = U$ for all $\gamma \in \langle H, K \rangle$, then $U = P_H$.*
- (ii) *For a nonempty clopen set $V \subset P_K$ such that $\gamma(V) = V$ for all $\gamma \in \langle H, K \rangle$, then $V = P_K$.*

Proof. The proof is the same as that of [10, Lemma 3.17]. \square

Therefore we have

Proposition 6.6 (cf. [10, Proposition 3.28]). *Let (H, K) be a Dye pair of subgroups of X_A . Then P_H and P_K are clopen sets of X_A such that*

$$P_H = (P_K)^\perp, \quad P_K = (P_H)^\perp \quad \text{and} \quad (H, K) = (\Gamma_{P_H}, \Gamma_{P_K}).$$

Proof. By Lemma 5.3, both P_H and P_K are clopen. Suppose that $P_H \cap P_K \neq \emptyset$. Since $\eta(\cap_{\zeta \in H} X_A^\zeta) = \cap_{\zeta \in H} X_A^\zeta$ for $\eta \in H$ and hence $\eta((\cap_{\zeta \in H} X_A^\zeta)^c) = (\cap_{\zeta \in H} X_A^\zeta)^c$, we have $\eta(P_H) = P_H$ for $\eta \in H$. Similarly $\kappa(P_K) = P_K$ for $\kappa \in K$. As P_H is $H^\perp (= K)$ -invariant and P_K is $K^\perp (= H)$ -invariant, the set $P_H \cap P_K$ is an H - and K -invariant clopen set. By Lemma 6.5, we have $P_H \cap P_K = P_H = P_K$. As in the proof of Lemma 5.3, the inclusion relations $H \subset \Gamma_{P_H}, K \subset \Gamma_{P_K}$ hold. Hence we have

$$\Gamma_{(P_H)^c} = \Gamma_{(P_H)^\perp} = (\Gamma_{P_H})^\perp \subset H^\perp = K \subset \Gamma_{P_K} = \Gamma_{P_H}$$

so that $(P_H)^c = \emptyset$. Therefore we have $P_H = P_K = X_A$. By Lemma 6.3, we have

$$[H]_c \cap [K]_c = \{\text{id}\}. \quad (6.5)$$

By Lemma 6.4, there exists $\alpha \in [H]_c$ such that the fixed point set X_A^α is not K -invariant. This implies $\alpha \notin H$. If $\alpha = \eta\kappa$ for some $\eta \in H$ and $\kappa \in K$, we see that $\eta^{-1}\alpha = \kappa$ belongs to $[H]_c \cap [K]_c$. Hence (6.5) implies $\eta^{-1}\alpha = \kappa = \text{id}$ and $\alpha = \eta \in H$, a contradiction, so that α does not belong to the subgroup HK . By condition (D4), one may find $\eta \in H \setminus \{\text{id}\}$ such that $\alpha\eta\alpha^{-1} \in K$. Since $\alpha\eta\alpha^{-1} \in [H]_c \cap K \subset [H]_c \cap [K]_c = \{\text{id}\}$, we obtain $\eta = \text{id}$ a contradiction. Thus we conclude

$$P_H \cap P_K = \emptyset.$$

This implies $P_H \subset (P_K)^c = (P_K)^\perp$. It then follows that

$$\Gamma_{P_H} \subset \Gamma_{(P_K)^\perp} = (\Gamma_{P_K})^\perp \subset K^\perp = H \subset \Gamma_{P_H}.$$

Therefore we have $\Gamma_{P_H} = H, P_H = (P_K)^\perp$ and similarly $\Gamma_{P_K} = K, P_K = (P_H)^\perp$. \square

7. MAIN RESULT AND ITS COROLLARIES

Let A, B be two irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I).

Lemma 7.1. *Suppose that there exists an isomorphism $\alpha : \Gamma_A \longrightarrow \Gamma_B$ of groups. Then for two subgroups H, K of Γ_A , the following two conditions are equivalent:*

- (i) (H, K) is a Dye pair of Γ_A .
- (ii) $(\alpha(H), \alpha(K))$ is a Dye pair of Γ_B .

Proof. We will show the implication (i) \implies (ii). Let (H, K) be a Dye pair of Γ_A . Since $\alpha : \Gamma_A \longrightarrow \Gamma_B$ is an isomorphism of groups, one knows that $(\alpha(H), \alpha(K))$ satisfies (D1), (D2) and (D3), and hence it is a strong commuting pair of Γ_B satisfying condition (D3). The conditions (D4) and (D5) are also determined by group structure, so that $(\alpha(H), \alpha(K))$ satisfies (D4) and (D5). Hence $(\alpha(H), \alpha(K))$ is a Dye pair of Γ_B . \square

An isomorphism $\alpha : \Gamma_A \longrightarrow \Gamma_B$ of groups is said to be *spatial* if there exists a homeomorphism $h : X_A \longrightarrow X_B$ such that $\alpha(\gamma) = h \circ \gamma \circ h^{-1}$ for $\gamma \in \Gamma_A$. We arrive at the main result of the paper.

Theorem 7.2. *Let A, B be two irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I). Then every group isomorphism $\alpha : \Gamma_A \longrightarrow \Gamma_B$ is spatial.*

Proof. The proof is achieved by constructing a Boolean isomorphism $\varphi : CO(X_A) \longrightarrow CO(X_B)$ satisfying

$$\alpha(\gamma) \circ \varphi = \varphi \circ \gamma, \quad \gamma \in \Gamma_A.$$

The construction of φ is due to Lemma 6.1, Propositions 6.6 and Lemma 7.1. The detailed proof is the same as the proof of [10, Theorem 4.2]. \square

Let us denote by \mathcal{O}_A the Cuntz-Krieger algebra for the matrix A , and \mathcal{D}_A its canonical maximal abelian subalgebra of \mathcal{O}_A ([5]).

Corollary 7.3. *Let A, B be two irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I). Then the following three conditions are equivalent:*

- (i) *The groups Γ_A and Γ_B are isomorphic as abstract groups.*
- (ii) *The one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*
- (iii) *There exists an isomorphism $\Psi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that $\Psi(\mathcal{D}_A) = \mathcal{D}_B$.*

Let N and M be the size of matrix A and that of B respectively. Denote by I_N and by I_M the identity matrix of size N and that of size M respectively. In [16], under the condition that $\det(A - I_N)\det(B - I_M) \geq 0$, an isomorphism between Cuntz-Krieger algebras induces an isomorphism between them which preserves their canonical maximal abelian subalgebras. Hence we have

Corollary 7.4. *Let A, B be two irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I). Suppose that $\det(A - I_N)\det(B - I_M) \geq 0$. Then the following two conditions are equivalent:*

- (i) *The groups Γ_A and Γ_B are isomorphic as abstract groups.*
- (ii) *The Cuntz-Krieger algebras \mathcal{O}_A and \mathcal{O}_B are isomorphic.*

By using classification theorem for Cuntz-Krieger algebras obtained by M. Rørdam [20](cf. [21]), one may classify the continuous full groups in terms of the underlying matrices under the determinant condition $\det(A - I_N)\det(B - I_M) \geq 0$ as follows:

Corollary 7.5. *Let A, B be two irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I). Suppose that $\det(A - I_N)\det(B - I_M) \geq 0$. The groups Γ_A and Γ_B are isomorphic as abstract groups if and only if there exists an isomorphism $\Phi : \mathbb{Z}^N / (A^t - I_N)\mathbb{Z}^N \longrightarrow \mathbb{Z}^M / (B^t - I_M)\mathbb{Z}^M$ such that $\Phi([1, \dots, 1]) = [1, \dots, 1]$.*

Therefore we know that there are many mutually nonisomorphic continuous full groups of one-sided topological Markov shifts such as the following corollary.

Corollary 7.6. *Let N, M be positive integers such that $N, M > 1$. Denote by $\Gamma_{[N]}$ and $\Gamma_{[M]}$ the continuous full groups of the one-sided full N -shift and M -shift respectively. Then $\Gamma_{[N]}$ and $\Gamma_{[M]}$ are isomorphic as abstract groups if and only if $N = M$.*

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